

# Maxwell's Equations and Shear Waves in the Vortex Sponge

Edward M. Kelly

Physics Department, California Polytechnic University, Pomona, CA 91768, USA

Z. Naturforsch. **45a**, 1367–1373 (1990); received July 26, 1990

The vortex sponge is a fluid substratum which is criss-crossed by innumerable fine hollow vortex tubes. Bending of tubes causes them to drift; the combination of drift and bending turns out to have effects mathematically identical to those of electric and magnetic fields. Thus, electromagnetism can be viewed mechanically, with Maxwell's curl equations governing the translation and rotation of the substratum. Shear and shear waves are illustrated by a plane shear wave.

## 1. Introduction

The vortex sponge (V.S.) was invented by J. Bernoulli and studied extensively by Kelvin and his contemporaries [1]. Various forms of the substratum known as the V.S. have been considered: vorticity may be distributed in an ideal fluid in a continuous or discontinuous way; in the latter case it may consist of a network of vortex tubes or filaments. If tubes, they may have hollow cores. In an interesting model developed by Winterberg [2], the substratum is an ideal fluid densely packed with tiny quantized vortex rings. Winterberg obtains analogies to electromagnetic phenomena and shows that waves can be propagated by the rings. The version of V.S. to be described below utilizes hollow vortex tubes closely intertwined to form a plenum somewhat resembling a large bowl filled with long loops, or even a single loop, of macaroni thoroughly stirred. The reason for closed loops instead of strands is that a vortex tube cannot end within the fluid and must be either reentrant or end on a boundary. If each hole in the macaroni is surrounded by an ideal fluid (incompressible, inviscid, structureless), instead of wheat, the fluid being in cyclic irrotational motion about each hole (core), an approximate picture of the V.S. results. This model has been treated previously [3, 4, 5] with the result that Maxwell's equations are shown to be characteristics of the substratum. In this paper some of the same ground is covered, but more simply than before. The treatment of shear and shear waves is new.

The parameters, or physical constants, of the V.S. are four in number:  $L$ , the length of vortex tube (here-

after, just "tube") per unit volume;  $\sigma$ , the cross-sectional area of the hollow core, taken as essentially circular;  $\kappa$ , the strength of circulation about the hollow core (the actual circulation being  $2\pi\kappa$ );  $\varrho$ , the mass density of the fluid without voids. In a sense,  $\varrho$  may be arbitrary or irrelevant, for if we assume that nothing but the fluid exists, an instantaneous change in  $\varrho$  everywhere produces an immediate change in pressure, the motion remaining unaltered. This implies that force is also an arbitrary or anthropomorphic concept (which need not affect its considerable usefulness). However, if it were possible to devise a model of matter, say an electron, as a conformation or substructure of the V.S.,  $\varrho$  could then, perhaps, be inferred in terms of the mass assigned to the electron. Thus,  $\varrho$  also takes on an anthropomorphic quality. Another mass density is that of the bulk medium; that is, the mass per unit volume of the V.S., including the voids of the tubes. This is slightly smaller than  $\varrho$ , but if the volume fraction of void is exceedingly tiny, as will be assumed here, the difference is small, so that  $\varrho$  is used below also as the density of the V.S.

Estimates of the parameters are necessarily crude at present. For good statistics, perhaps  $10^6$  tubes might be distributed over a representative section of a proton,  $10^{-30} \text{ m}^2$ , to give a density of  $10^{36}$  tubes per  $\text{m}^2$  and a mean distance between tubes of  $10^{-18} \text{ m}$ . The core diameter of a tube might be a thousandth of this, to give a  $\sigma$  of  $10^{-42} \text{ m}^2$ . These may be gross underestimates; Winterberg (private communication) has pointed out that to avoid conflicts with special relativity the tube density may be many orders of magnitude greater than  $10^6$  per (Fermi)<sup>2</sup>. Such an increase would have no effect on the results in this paper. No estimate of  $\kappa$  appears to be possible at present. It can be an average value for many different tubes, or, to use Oc-

Reprint requests to E. M. Kelly, 8115 Hillside Road, Alta Loma, CA 91701, USA.

0932-0784 / 90 / 1100-1367 \$ 01.30/0. – Please order a reprint rather than making your own copy.



Dieses Werk wurde im Jahr 2013 vom Verlag Zeitschrift für Naturforschung in Zusammenarbeit mit der Max-Planck-Gesellschaft zur Förderung der Wissenschaften e.V. digitalisiert und unter folgender Lizenz veröffentlicht: Creative Commons Namensnennung-Keine Bearbeitung 3.0 Deutschland Lizenz.

Zum 01.01.2015 ist eine Anpassung der Lizenzbedingungen (Entfall der Creative Commons Lizenzbedingung „Keine Bearbeitung“) beabsichtigt, um eine Nachnutzung auch im Rahmen zukünftiger wissenschaftlicher Nutzungsformen zu ermöglichen.

This work has been digitalized and published in 2013 by Verlag Zeitschrift für Naturforschung in cooperation with the Max Planck Society for the Advancement of Science under a Creative Commons Attribution-NoDerivs 3.0 Germany License.

On 01.01.2015 it is planned to change the License Conditions (the removal of the Creative Commons License condition "no derivative works"). This is to allow reuse in the area of future scientific usage.

cam's razor, it can be the same for all tubes. This would be the case if the plenum contains just a single vortex ring twisted about on itself innumerable times to form a sort of ball of string. The suggestion here of a precursor to Planck's constant (of which there is apparently only one!) is inescapable, although the connection has not yet been identified. Dimensionally,  $\kappa$  cannot be Planck's constant, since its mks units are  $\text{m}^2 \text{s}^{-1}$  rather than  $\text{kg m}^2 \text{s}^{-1}$ .

The above guesses give no clue to the value of the mass or energy density of the V.S. In the nineteenth century frequent references were made to matter and ether, the former being regarded as solid and the latter as an extremely tenuous or rare substance through which matter moved like a submarine. Nothing like that is contemplated here. Matter is imagined as a structure (yet to be determined) in the V.S. itself. If so, it is possible that its presence alters the V.S. only slightly; that is, matter may be a small perturbation of the V.S. Unlike the 19th century viewpoint, matter is considered rare and ether dense! The density of a proton is of the order of  $10^{18} \text{ kg m}^{-3}$ ; if it involves on the average only  $10^{-6}$  part of the V.S. which it occupies, the density of the V.S. would be  $10^{24} \text{ kg m}^{-3}$ , or  $10^{21}$  times as dense as water. The passage of an object through space is regarded as a propagation of energy and form (a wave), rather than a motion of a solid through liquid. Only the form, not the substance, moves with the object. Arguments against the ether concept on the basis that planets would be slowed down by moving through it are without merit from this point of view.

The notion of great ether-density conforms to current thinking about substratum densities suggested by zero-point fluctuations. These are sometimes even more drastic than the estimate of  $10^{24} \text{ kg m}^{-3}$  above. Division of the Planck mass by the cube of the Planck length yields  $10^{97} \text{ kg m}^{-3}$ .

The unpopularity of ether theories in this century was a consequence of special relativity, which was erroneously thought to be incompatible with them. The rod contraction-clock retardation ether theory which yields the Lorentz Transformations was evidently not developed until about 1909 [6]. The fact that Maxwell's vacuum equations have been shown to govern the mechanics of the V.S. indicates that in spite of its Newtonian postulates, the V.S. has relativistic properties. Winterberg has pointed out that if the ether is the cause of relativistic effects it should not, itself, be subject to the laws of relativity [7].

## 2. Properties of Vortex Tubes

The vortex tubes dealt with here are hollow cores with ideal fluid in cyclic irrotational motion around them. The circulation constant,  $\kappa$ , is defined such that the circulation around each tube is  $2\pi\kappa$ . For simplicity  $\kappa$ , which will also be called the "spin", is assumed to be the same for all tubes; if this were strictly true,  $\kappa$  would then be a natural quantum for the substratum. As the tube moves about, the circulation is carried with it (Kelvin's circulation theorem: the circulation in any circuit which moves with an inviscid fluid of constant density is constant).

A straight vortex tube induces no motion in itself, but if there is a transverse current past the tube, the fluid velocity on one side is greater than on the opposite. This results (Bernoulli's Theorem) in a generally lower pressure on the high-velocity side, with a corresponding force on the tube. This force is illustrated by an airfoil with circulation in a wind-tunnel, with the corresponding "lift".

The lift force per unit length of tube is  $2\pi\rho\kappa v$ , where  $v$  is the current speed. Conversely, if there is no current, but the tube is moving with speed  $v$ , the tube exerts force  $2\pi\rho\kappa v$  per unit length on the fluid.

If a tube, initially stationary, is subjected to a constant force, it accelerates first in the direction of the force, but as the velocity builds up, the sidewise force,  $2\pi\rho\kappa v$ , causes it to move sidewise. The ultimate effect, if we neglect the oscillatory part of the motion, is a sidewise displacement which is labelled  $\xi$ . The quantities  $\xi$  and  $\dot{\xi}$  enter prominently into the following discussion in relation to the electric vectors  $\mathbf{E}$  and  $\dot{\mathbf{E}}$ .

The circulation constant  $\kappa$ , can also be regarded as a vector  $\boldsymbol{\kappa}$  of the same magnitude in the direction given by the familiar right-hand rule: grasp the tube in the right hand with the fingers curled in the sense of the spinning fluid; the extended thumb is then in the direction of  $\boldsymbol{\kappa}$ . A stationary tube in a transverse current of velocity  $\mathbf{v}$  has a lift force per unit length of  $2\pi\rho\boldsymbol{\kappa} \times \mathbf{v}$ . If a velocity of  $-\mathbf{v}$  is superimposed everywhere, the current is annihilated and the tube is drifting transversely with velocity  $-\mathbf{v}$ . The thrust of the tube on the fluid is the same as before and, setting  $-\mathbf{v} = \dot{\boldsymbol{\xi}}$ , is  $2\pi\rho\dot{\boldsymbol{\xi}} \times \boldsymbol{\kappa}$ . Further details are available in [8], p. 251 and 351 ff.

When a straight tube is inclined to a plane (a geometrical surface which does not interfere with fluid motion), momentum is carried across the plane. Suppose the plane to be the  $xz$  plane, Fig. 4, with  $x$  to the

right,  $z$  out of the paper and  $y$  toward the top of the page. Let  $\mathbf{x}$  be upward in the plane of the paper ( $x, y$ ) and inclined toward  $x$ . It is apparent that, in the half-plane  $(x, 0, +z)$ , momentum crosses the plane downward with a  $+x$  component, while in the half-plane  $(x, 0, -z)$  the fluid crosses in the upward direction with a  $-x$  component. That is, positive  $x$ -momentum enters region  $R$  below the  $xz$ -plane and negative  $x$ -momentum leaves  $R$ . Both effects constitute positive  $x$ -momentum entering  $R$ . That is, an inclined tube generates a shear on a surface in the sense of the inclination. It is obvious that reversing the sign of  $\mathbf{x}$  produces the same shear, so that if the neutral V.S. ("neutral" meaning that the tube directions are random, so that statistically the spins cancel) undergoes a shearing strain, a shear force results.

If we consider a slab of the substratum intersected by tubes which are initially straight, no net force results from the straight tubes. The shear force resulting from a particular tube on, say, the top of the slab, is cancelled by an equal but opposite force on the bottom of the slab. This equality is upset if the medium is displaced so that the tubes become curved to produce a net force. This force is proportional to the curvature and in the direction of concavity.

A point about tube curvature needs clarification: in any non-linear medium displacement, tubes that were initially straight are bent. This does not necessarily produce a net thrust on the fluid, since some tubes may be bent one way and some another. A displacement that carries a plane into a saddle-shaped surface is an example. The thrust on a fluid element occurs from the vector sum of the thrusts of all tubes in the element. It is shown in the next section that this vector sum is zero when the displacement  $\mathbf{D}$  is the gradient of a scalar.

Another property of bent tubes is that they drift laterally; one way to see this is to think of a segment of a smoke ring. The ring advances in the same direction as the fluid flowing through the ring. This is in the direction of  $\mathbf{x} \times \mathbf{K}$ , where  $\mathbf{x}$  is the spin vector and  $\mathbf{K}$  is the vector curvature, in this case directed toward the center of the ring. A more fundamental reason can be demonstrated by considering a solid rod in the fluid with circulation  $2\pi\kappa$ . If the rod becomes curved, the fluid velocity is greater on the concave side of the curve than on the convex side; the pressure on the concave side is consequently smaller than on the convex side. If the rod is suddenly annihilated so as to produce a hollow core, the pressure gradient acceler-

ates the tube in the direction of concavity (if it is a ring, the ring contracts). The velocity arising from this acceleration generates a lift force which causes the tube to drift in the direction of  $\mathbf{x} \times \mathbf{K}$ . For example, if the tube in Fig. 2 is concave toward  $+z$ , it will drift to the right, as in the figure. If the curved tube starts from rest the drift is not uniform; the tube starts moving toward  $+z$  but as it acquires speed, the lift force causes it to move toward the right. The speed increases until the lift force exceeds the curvature force and the tube returns to its original  $z$ -position, but with a translation to the right. This process then repeats indefinitely. If we ignore the  $z$ -motion, the net effect is a drift toward the right. The tube, in this special case, traces out a cycloid in the  $xz$  plane; in more general cases, the trace is a trochoid. A more detailed analysis is given in [8], p. 250.

Since a tube with circulation drifting with velocity  $v$  exerts a force  $2\pi\kappa\varrho v$  per unit length on the fluid, it might appear that drifting tubes will accelerate the fluid. However, a void cannot exert a force on the fluid. The  $2\pi\kappa\varrho v$  force is the force exerted by a rigid core, not a void. Nevertheless, the force exists, but for another reason which is now explored.

### 3. Bending of Tubes Under a Medium Displacement

Under a bulk displacement  $\mathbf{D}(x, y, z)$ , bending of tubes may occur. If there are no singularities (e.g., sources) in the neighborhood of the origin,  $\mathbf{D}$  will have a Maclauren's expansion that can be expressed as the sum of vectors whose magnitudes are polynomials of increasing degree. Constant and linear vector displacements, although shifting and tilting tubes, do not bend them, so that the first two terms of the expanded form of  $\mathbf{D}$  can be ignored for the present purpose. Terms of the third and higher degrees can also be ignored near the origin, since the two differentiations involved in calculating the curvature leave terms containing  $x$ ,  $y$  or  $z$  which vanish at 0. Thus, for calculating the curvature of a tube at 0, it is sufficient to write

$$D_x = a_1 x^2 + a_2 y^2 + a_3 z^2 + a_4 xy + a_5 xz + a_6 yz, \quad (1)$$

with similar expressions, with  $b$  and  $c$  coefficients, for  $D_y$  and  $D_z$ . The curvature of a curve is  $d^2\mathbf{r}/ds^2$ , where  $\mathbf{r}$  is the position vector of a point  $P$  on the curve and  $s$  is its distance along the curve measured from a fiducial point, in this case, 0. For a straight tube through 0 having direction angles  $\alpha$ ,  $\beta$ ,  $\gamma$ , before

the displacement,  $P$  has the coordinates  $x = s \cos \alpha$ ,  $y = s \cos \beta$ ,  $z = s \cos \gamma$ . After the displacement the transformed point  $P'$  having position vector  $\mathbf{r}$  has coordinates  $r_x = s \cos \alpha + D_x$ ,  $r_y = s \cos \beta + D_y$ ,  $r_z = s \cos \gamma + D_z$ . The  $x$ -component of curvature of the transformed tube evaluated at 0 is then

$$\begin{aligned} K_x &= (d^2 r_x / ds^2) = (d^2 / ds^2)(s \cos \alpha + a_1 s^2 \cos^2 \alpha + \cdots \\ &\quad + a_5 s^2 \cos \alpha \cos \gamma + a_6 s^2 \cos \beta \cos \gamma) \\ &= 2(a_1 \cos^2 \alpha + a_2 \cos^2 \beta + \cdots \\ &\quad + a_5 \cos \alpha \cos \gamma + a_6 \cos \beta \cos \gamma). \end{aligned}$$

In the neutral V.S., tube directions are distributed equally in all directions, so that the mean values of terms like  $\cos^2 \alpha$  are given by

$$\int_0^{2\pi} \int_0^\pi \cos^2 \alpha \sin \alpha \, d\alpha \, d\varphi \bigg/ \int_0^{2\pi} \int_0^\pi \sin \alpha \, d\alpha \, d\varphi = 1/3,$$

while the mean values of terms like  $\cos \alpha \cos \beta$  are zero from symmetry. Thus, the mean tube curvature is

$$\langle K_x \rangle = (2/3)(a_1 + a_2 + a_3),$$

with similar expressions with  $b$  and  $c$  coefficients for  $\langle K_y \rangle$  and  $\langle K_z \rangle$ . The components of  $\langle \mathbf{K} \rangle$  are the same as those of  $1/3 \nabla^2 \mathbf{D}$ , as can be seen from (1), so that

$$\langle \mathbf{K} \rangle = \frac{1}{3} \nabla^2 \mathbf{D} = -\frac{1}{3} \nabla \times (\nabla \times \mathbf{D}),$$

the last equation deriving from the well-known vector identity when  $\nabla(\nabla \cdot \mathbf{D}) = 0$ , true in this case since by hypothesis there are no sources near the origin. The thrust per unit volume is proportional to the mean curvature, so that the equation of motion, provided that no bulk pressure gradients are present, is

$$-G \nabla \times (\nabla \times \mathbf{D}) = \varrho \ddot{\mathbf{D}}, \quad (2)$$

where  $G$  is the constant of proportionality. When  $\mathbf{D}$  is the gradient of a scalar, the thrust is zero, even though tubes may be curved.

The substitutions

$$\mathbf{B} \propto \nabla \times \mathbf{D} \quad \text{and} \quad \mathbf{E} \propto -\ddot{\mathbf{D}} \quad (3)$$

change the form of (2) to

$$\nabla \times \mathbf{B} = k_1 \dot{\mathbf{E}}, \quad (4)$$

where  $k_1$  is a constant. This has the form of one of Maxwell's curl equations.

The relations among the various vectors is illustrated by a simplified case in Figure 1. The displacement  $\mathbf{D}$  diminishes upward at an increasing rate so that  $\mathbf{B}$ , the curl of  $\mathbf{D}$ , is in the  $+z$  direction, at a

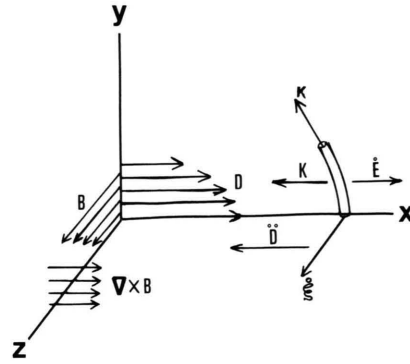


Fig. 1. Relations among the electromagnetic vectors.

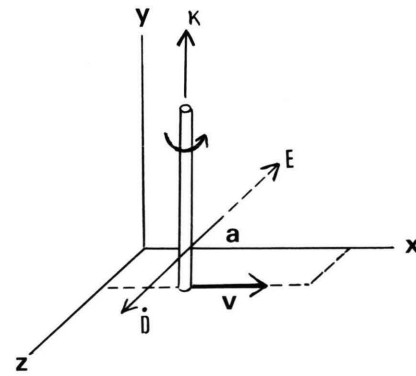


Fig. 2. Tube drift and thrust to develop an electric field.

rate increasing with  $y$ . From the figure, as well as the definition of curl, it is clear that  $\mathbf{B}$  represents a shearing twist or rotation of the medium. The curl of  $\mathbf{B}$  is seen to be in the  $+x$  direction. Since the tubes, represented by the single tube section at the right of Fig. 1, are bent so as to be concave toward the left (to greater or lesser degree, depending upon their orientation) they will drift toward  $+z$  as shown if the spin is upward or  $-z$  (not shown) if the spin is downward. In either case, the thrust of the tubes is toward the left, this being the direction of the mean curvature  $\mathbf{K}$  and the acceleration  $\ddot{\mathbf{D}}$  of the medium.

In (3),  $\mathbf{E}$  was defined as  $-\ddot{\mathbf{D}}$ . There is a simple interpretation of this. In Fig. 2, a tube section of length  $l$  drifts uniformly from its original position 0 to  $a$ . While drifting it exerts a force  $2\pi\kappa\varrho vl$  on the fluid. It takes time  $a/v$  to get to  $a$ , where it comes to rest. During its travel it delivers an impulse  $2\pi\kappa\varrho vl(a/v) = 2\pi\kappa\varrho A$  to the fluid, where  $A$  is the area swept out by the tube. Similar action is taking place in the neighborhood of the section  $l$ ; this could conceivably affect the fluid



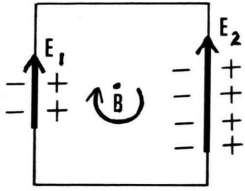
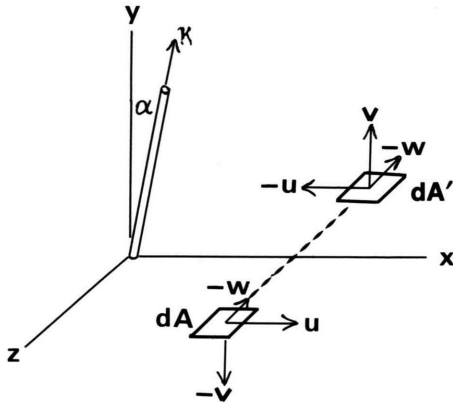
Fig. 3. Curl of  $E$  related to angular velocity,  $\dot{B}$ .

Fig. 4. Origin of shear stress from tilted tubes.

near  $l$  by generating pressure gradients. If such gradients are negligible a volume element of the fluid would be affected only by tube sections within itself. The force per unit volume from drifting tubes may be defined as the thrust  $T$ ; it is proportional to the mean vector curvature  $K$ .

The impulse per unit volume is  $2\pi\kappa\varrho A'$ , where  $A'$  is the total projected area swept out by all the tube sections in the volume, the projection being in the direction of an observer looking along the thrust vector. Now define  $E \propto \varrho A'$  so that the impulse is proportional to  $E$ . But if the fluid was at rest initially, its momentum after the tubes stop drifting (recalling the assumption of no pressure gradients) is  $\varrho \dot{D}$ . Therefore  $E \propto \pm \dot{D}$ , the sign being conventional.

#### 4. Time Variation of $B$

The medium velocity  $\dot{D}$  illustrated in Fig. 2 is proportional to  $a$  and therefore to  $E$ , so that  $E$  can be interpreted in terms of the lateral mean translation of tubes. This can result in rotatory motion of the bulk

medium. A simplified example of this is shown in Fig. 3, where  $E$  increases toward the right. The signs symbolize the intersections of tubes with the plane of the paper, + indicates counterclockwise and - clockwise spin. It is apparent that  $E_2$ , corresponding to greater lateral tube translation, will contribute more negative tubes to the area  $dy dx$  than are removed by  $E_1$ . At the same time more positive tubes are removed from the area by  $E_2$  than are added by  $E_1$ . For both reasons a net negative spin density arises in the previously neutral V.S.

The area will then have net bulk circulation so that it spins like a body under a changing shear or in rotation, or both. Since, by definition,  $B \propto \nabla \times D$ , it represents at any instant a shear or angular displacement, so that  $\dot{B}$ , like  $\nabla \times E$ , is a measure of angular velocity. Together with (4), we now have

$$\nabla \times B = k_1 \dot{E}, \quad \nabla \times E = -k_2 \dot{B}. \quad (5)$$

The explicit negative sign emphasizes that  $\nabla \times E$  and  $\dot{B}$  are in opposite senses. This is evident from Fig. 3, where  $\nabla \times E$  is counterclockwise and  $\dot{B}$  clockwise.

Equations (5) are Maxwell's equations, so that the V.S. has electromagnetic properties.

#### 5. Normal and Tangential Stresses

Consider a straight tube, Fig. 4, whose axis, in the  $xy$  plane, is inclined toward the  $x$ -axis at a small angle  $\alpha$  with the  $y$ -axis. When the spin (circulation) is in the upward sense, fluid is entering region  $R$  below the  $xz$  plane through area element  $dA$  in the  $xz$  plane with velocity components  $u$ ,  $-v$ ,  $-w$ , where  $u$ ,  $v$ ,  $w$  are positive numbers. At  $dA'$ , the mirror image of  $dA$  in the  $xy$  plane, fluid is leaving  $R$  with velocity components  $-u$ ,  $v$ ,  $-w$ . The fact that  $w$  has the same sign at both  $dA$  and  $dA'$  means that momentum enters  $R$  at  $dA$  with a  $-z$  component while an equal amount leaves  $R$  across  $dA'$ , also with a  $-z$  component. The forces associated with these momentum transports therefore cancel, so that  $z$  components may be ignored for the purpose at hand. By contrast, the  $x$  and  $y$  components change sign from  $dA$  to  $dA'$  so that the resultant force on  $R$  from momentum transport over  $dA$  and  $dA'$  has a positive  $x$ -component and a negative  $y$ -component. When pressure  $p$  is taken into account, the force components on  $R$  associated with  $dA$  and  $dA'$  are

$$dF_x = 2\varrho v u dA; \quad dF_y = -2p dA - 2\varrho v^2 dA.$$

For small values of  $\alpha$ ,  $v$  may be taken as  $\pm u\alpha$ , since  $v=0$  for  $\alpha=0$ , so that

$$dF_x = 2qu^2\alpha dA; \quad dF_y = -2p dA, \quad (6)$$

when the higher order term  $2qv^2 dA = 2qu^2\alpha^2 dA$  is dropped.

For a single tube the force on  $R$  is infinite because the fluid velocity varies as the inverse first power of the distance from the tube axis, but in the V.S. this power law does not hold because of the presence of other tubes with opposite spins. In such a case there will be finite normal and tangential stresses on  $R$ , the latter being in the  $+x$  direction. No work is transmitted to  $R$  because as much kinetic energy is leaving  $R$  over  $dA'$  as enter at  $dA$  and the power density term  $pv$  at  $dA'$  is cancelled by  $-pv$  at  $Q$ . If the spin is reversed  $dF_x$  and  $dF_y$  are unchanged so that in the neutral V.S. where the spins balance statistically, a shear strain will induce a shear stress.

When the tube is moving to the right with speed  $U$ , pressures at  $dA$  and  $dA'$  are altered in the same way as if the tube remains stationary and a uniform stream toward the left of speed  $U$  is superimposed. This avoids the slightly more complicated case for which the velocity potential  $\phi$  is time-dependent, requiring the term  $\rho \partial \phi / \partial t$  in the pressure equation ([2], p. 89). Then Bernoulli's equation is

$$p = p_0 - \frac{1}{2} \rho [(u-U)^2 + v^2 + w^2] + \frac{1}{2} \rho U^2 \\ = p_0 - \frac{1}{2} \rho (u^2 + v^2 + w^2) + \rho u U, \quad (7)$$

where  $p_0$  is the pressure at infinity where the parenthesis is zero. Only the last term differs from that for  $U=0$ .

The power expended on  $R$  at  $dA$  is  $pv dA$  while at  $dA'$  it is  $-p'v dA$ , so that for both  $dA$  and  $dA'$ , the net power expended on

$$R = (p - p') dA \cdot v = 2quU v dA = (2qu^2\alpha dA) U.$$

This is seen from (6) to be the same as  $U dF_x$ . Since the net kinetic energy transport into  $R$  remains at zero,  $U dF_x$  measures the total power transport across  $dA$  and  $dA'$ . If the shear stress on  $R$  is  $S$ , then the power density, not unexpectedly, is  $SU$ .

A point of interest is that  $S$  and  $U$  did not appear in the derivation as such. That is, to get  $U dF_x$  we did not multiply  $U$  by  $2qu^2\alpha dA$ , but rather  $v$  by  $p dA$ . Thus,  $S$  times  $U$ , although a correct measure of the power, is not its *cause*; instead, the stresses and the power flow are concomitant effects of the tube motions.

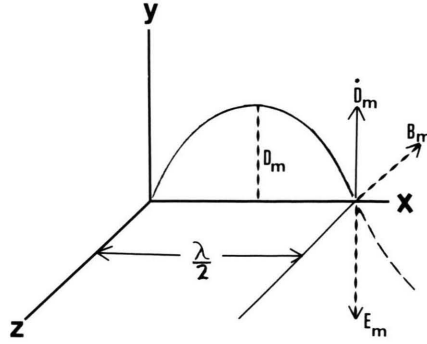


Fig. 5. Relations among electric, magnetic and hydrodynamic vectors in a plane wave.

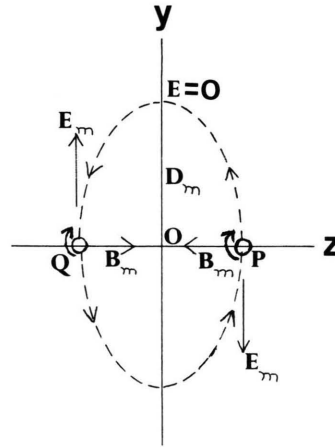


Fig. 6. Trace of a tube cross-section during passage of a plane wave.

As mentioned above, the stresses on  $R$ , whether  $U$  is zero or not, are independent of the direction of spin, so that  $S$  is always in the direction of the tilt of the tubes.

## 6. Shear Waves

In the V.S. the tube orientations are randomly distributed, but one can think of them as being resolved into  $x$ ,  $y$ ,  $z$  components to give an idealized V.S. in which a third of the tubes are equally parallel and antiparallel to each axis. A shear in, say, the  $y$ -direction would then tilt only those components parallel to the  $x$ -axis. Suppose the medium displacement to be given by  $\xi(x)$ ; tubes would then be tilted through the (small) angle  $\xi'(x)$  at the plane  $x = \text{const}$ , where the prime denotes a partial derivative. The tilt for positive

$\xi'(x)$  is downward relative to the left-hand side of the slab between  $x$  and  $x + dx$ . The shear stress on the slab at  $x$  is then  $-\eta \xi'(x)$ , where  $\eta$  is the shear modulus. At  $x + dx$  the stress on the slab is upward of magnitude  $\eta \xi'(x + dx) = \eta [\xi'(x) + \eta dx \xi''(x)]$ , so that the net force per unit volume on the slab is  $\eta dx \xi''(x)/dx = \eta \xi''(x)$ . The thrust per unit volume originating from the tube curvature is  $G \nabla^2 \mathbf{D}$ , where  $G$  is a constant and  $\mathbf{D}$  the (divergenceless) medium displacement. In the present case this reduces to  $G \xi''(x)$ , so that the shear modulus  $\eta = G$ .

A single tube making a small angle  $\alpha$  with the normal to a surface can be said to exert a shearing force at the surface of amount  $(3G/L)\alpha$ , where  $L$  is the tube density (tube length per unit volume, or, equally, the number of tubes which intersect unit area). The factor 3 enters if one takes the idealized (resolved) form of the V.S. where the tubes are parallel to the axes and uniformly arranged with spins alternating so as to form a neutral substratum. The quantity  $G/L$  contains factors which are the circulation (spin) constant, fluid density and drift coefficient (Ref. [1], App. A, p. 132). One can think of the tube as "controlling" a cylindrical region of fluid circulating about the tube axis. The

"cylinder" is, of course, a statistical abstraction, since the tube in the environment of the V.S. cannot be a straight, unchanging entity.

Figure 5 shows a plane sine wave, a shear wave of amplitude  $D$ , progressing in the  $x$ -direction. From the relations  $\mathbf{B} \propto \nabla \times \mathbf{D}$ ,  $\mathbf{E} \propto -\dot{\mathbf{D}}$ , it is easily seen that maximum values (subscript m) of  $B$  and  $E$  occur at zeros of  $D$ . When  $D$  is a maximum,  $E$  and  $B$  are zero. Figure 6 shows a cross-section of a single tube which, without the wave, is parallel to the  $x$ -axis. The cross-section may be considered to lie in the plane  $x = \lambda/2$  of Figure 5. Since  $E$  is a maximum when  $D = 0$ , the tube will be displaced to a point  $P$  on the  $z$ -axis. As  $D$  increases the tube develops downward concavity and, with clockwise spin as shown, will drift toward the  $y$ -axis, reaching it when  $D$  reaches maximum. The concavity is now a maximum, accelerating the fluid downward; the tube continues to drift until it reaches  $Q$ . The process continues, the tube cross-section tracing out an elliptical path in the  $yz$ -plane. The points  $Q$  and  $P$  represent maximum  $E$ , upward at  $Q$  and downward at  $P$ . Maximum tilt also occurs here, the twist due to shear being proportional to  $B_m$  in the sense shown in Figure 6.

[1] E. W. Whittaker, *A History of the Theories of Aether and Electricity*, vol. 1, Thomas Nelson and Sons Ltd., London 1951, pp. 95, 295.

[1a] A survey of the history of the V.S. is available in [1] from the following pages: 95, 143, 295, 296, 299, 301. For original sources, consult:

[1b] J. Bernoulli, *Recueil des pièces qui ont remportés les prix de l'Acad.*, tome iii (1752).

[1c] J. MacCullagh, *Trans. Roy. Irish Acad.*, xxi (1848), p. 17. Also, *MacCullagh's Coll. Works*, p. 145.

[1d] W. Thompson, *Brit. Assoc. Rep.* (1880), p. 473; *ibid.* (1885), p. 930. Also, *Kelvin's Math. and Physics Papers*, Vol. IV, pp. 1–230, C. J. Clay and Sons, Cambridge University Press Warehouse, London 1890.

[1e] G. Fitzgerald, *Scientific Writings*, pp. 254, 472, 484.

[1f] W. M. Hicks, *Phil. Trans. Roy. Soc.*, clxxv (1883), p. 161; clxxvi (1885), p. 725; cxcii (1898), p. 33.

[1g] R. V. Hartley, *Bell System Tech. Jour.* xxix (1950), p. 350.

[1h] M. C. Duffy, *Proc. Brit. Soc. Phil. Sci.*, London (1990), p. 136. A general review and excellent bibliography on the V.S.

[2] F. Winterberg, *Z. Naturforsch.* **43a**, 1131 (1988).

[3] E. M. Kelly, *Amer. J. Phys.* **31**, 785 (1963).

[4] E. M. Kelly, *Amer. J. Phys.* **32**, 657 (1964).

[5] E. M. Kelly, *Nuovo Cim.* **32B**, 117 (1976).

[6] E. Ehrlichson, *Amer. J. Phys.* **41**, 1068 (1973).

[7] F. Winterberg, *Z. Naturforsch.* **42a**, 1440 (1987).

[8] L. M. Milne-Thomson, *Theoretical Hydrodynamics*, 5th ed., The Macmillan Press Ltd., London 1979.